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FUNDAMENTAL THEOREMS ON STABILITY OF A PROCESS IN A PRESCRIBED TIME INTERVAL*

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The concept of stability of a process on a prescribed time interval was formulated in /1/. General theorems establishing the stability and instability conditions for the unperturbed motion (the trivial solution of the system of equations of perturbed motion) in a prescribed time interval were announced in /2/. Brief proofs of these theorems are presented below.

1. The concept of stability in a prescribed time interval is introduced as follows /1/. Let $\omega(t)$ be some prescribed positive function and let $\omega(t_0)=\omega_0; K_\Delta{}^\omega$ be the class of $n\times n$ matrices $G(t)=(G_1, G_2, \ldots, G_n)$ over the complex number field, satisfying, on a prescribed interval $\Delta=[t_0, T)$, where T is a number exceeding t_0 or is ∞, the conditions $\det G\neq 0$; let the Hermitian norm of the columns $G_j\ (j=1,2,\ldots,n)$ coincide with $\omega(t)$, i.e., $(G_j, G_j)^{1/2}=\omega(t)$, $t\in\Delta$. Assuming that the deviations of the perturbed process' parameters from those of the unperturbed process are represented by the vector-valued function $x(t)$ (an $n\times 1$ column-matrix), we define the stability of a process in interval Δ.

Definition. If in a prescribed class $K_\Delta{}^\omega$ there exists a matrix G coinciding at the initial instant $t=t_0$ with a prescribed constant matrix G_0 of class $K_\Delta{}^\omega$, such that for a sufficiently small $\rho>0$ the perturbation of the process whose initial value $x_0=x(t_0)$ satisfies the condition

$$(G_0{}^{-1}x_0, G_0{}^{-1}x_0)\leqslant\rho^2 \tag{1.1}$$

in the interval $\Delta=[t_0, T)$ satisfies the condition

$$(G^{-1}x, G^{-1}x)\leqslant\rho^2 \tag{1.2}$$

then the unperturbed process in stable in the prescribed interval Δ. Otherwise, it is unstable.

Together with this basic definition of stability in a prescribed time interval there have also been formulated /1/ the concepts of stability uniform on Δ, stability on an unbounded interval $[t_0, \infty)$, asymptotic stability on an unbounded interval, etc.

Note. This concept of stability in a prescribed time interval is, in some sense, a generalization of the stability concepts introduced earlier by other authors. Thus, when $\omega(t)=\mathrm{const}$ our definition coincides with Kamenkov's definition /3/ of motion stability on a finite time interval. If we reckon that inequality (1.1) prescribes a domain of initial perturbations, while inequality (1.2) prescribes a domain of admissible perturbations on interval Δ, then our definition of stability coincides with the concept of practical stability /4,5/. With an insignificant modification of our definition of stability we can get more complex concepts of stability on a finite interval, such as quasi-stability and contractive practical stability, asymptotic stability on a prescribed time interval in Krasovskii's sense /7/, etc.

2. We consider dynamic systems whose perturbed motion is represented by

$$dx/dt=f(t, x, g),\ f(t, 0, 0)\equiv 0 \tag{2.1}$$

where $f(t, x, g)$ is a vector-valued function satisfying the existence and uniqueness conditions for the solution of the Cauchy problem in domain $I_0\times D_x\times D_g$ (D_x and D_g are like open sets in the corresponding vector spaces, and $I_0\subset[0\leqslant t<\infty)$). We assume that the vector-valued function g in (2.1) is some known or unknown vector-valued function of time t and of the phase (state) vector x, bounded by some condition

$$g(t)\subset\pi(t)\subset D_g\quad(t\in I_0) \tag{2.2}$$

*Prikl.Matem.Mekhan.,45,No.3,412-418,1981 where $\pi(t)$ is some domain of possible or admissible values of the perturbed forces, known or subject to definition. General theorems establishing the stability and instability conditions for the process (for the trivial solution of Eq. (2.1)) were announced in /2/. Below we briefly present the proofs of these theorems.

As is well known, an arbitrary rectangular $m \times n$ matrix of rank r can be given as a product of two matrices B and C of dimensions $m \times r$ and $r \times n$, respectively. The following lemmas hold.

Lemma 2.1. In order that a Hermitian matrix (of order n) be representable as

$$A = B^*B \tag{2.3}$$

where B is some, in general, rectangular $m \times n$ matrix, it is necessary and sufficient that it not have negative eigenvalues.

Lemma 2.2. Let $A(t)$ be an nth-order Hermitian matrix admitting of expansion (2.3) on the interval $t_0 \leqslant t < T$, where B is a square matrix of the same order n, and

$$\sup |B(t)| < \infty, \quad \det |B(t)| \geqslant a > 0, \ t \in [t_0, T)$$

Then the eigenvalues $\rho_i(t)$ of matrix A are bounded from below by a positive constant on the interval $[t_0, T)$.

Lemma 2.3. Let A and B be Hermitian matrices related by

$$A = HBH^*, H = (h_1, h_2, \ldots, h_n), h_j^*, h_j = \alpha^2 \ (j = 1, 2, \ldots, n)$$

(H is a square matrix); let at least one of the following two conditions be fulfilled: a) $h_i^*h_j = 0$ $(i \neq j; i, j = 1, 2, \ldots, n)$ or b) B is a diagonal matrix. Then

$$\operatorname{Tr} A = \alpha^2 \operatorname{Tr} B$$

The lemma's proof follows from

$$\operatorname{Tr} A = \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{r=1}^{n} h_{jr} b_{rk} \bar{h}_{jk}$$

In the special case when B is the unit matrix, i.e., $A = HH^*$, $\operatorname{Tr} A = n\alpha^2$.

Lemma 2.4. Let Λ be a real diagonal matrix with diagonal elements $\mu_1, \mu_2, \ldots, \mu_n$ satisfying the condition

$$\mu_i \neq \frac{1}{n} \operatorname{Tr} \Lambda = \frac{1}{n} \sum_{j=1}^{n} \mu_j \quad (i = 1, 2, \ldots, n) \tag{2.4}$$

Then the expansion

$$\Lambda = RR^* \tag{2.5}$$

holds, where R is an nth-order square matrix whose columns have the like Hermitian norm

$$\sqrt{R_j^* R_j} = \alpha^2 \quad (j = 1, 2, \ldots, n) \tag{2.6}$$

$$\alpha = \sqrt{\frac{1}{n} \operatorname{Tr} \Lambda}, \quad \operatorname{rank} R = \operatorname{rank} \Lambda$$

and, if Λ is a function of t, continuous on $[t_0, T)$ and l times differentiable $(l = 1, 2, \ldots)$, then $R(t)$ is, respectively, continuous and l times differentiable on $[t_0, T)$.

The matrix

$$R = \sqrt{\Lambda} V \tag{2.7}$$

where V is an arbitrary unitary matrix, satisfies (2.5). By Lemma 2.3 we have $\operatorname{Tr} \Lambda = n\alpha^2$. Allowing as well for (2.6), we obtain (2.5). Let us show that a unitary, and even orthogonal, matrix V does indeed exist such that the matrix R of form (2.7) satisfies (2.6). Relations (2.5)—(2.7) lead to the following equalities relative to the columns V_j of matrix V:

$$V_j' \left(\Lambda - \frac{1}{n} \operatorname{Tr} \Lambda E_n\right) V_j = 0 \tag{2.8}$$

$$V_i' V_j = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{2.9}$$

In an n-dimensional Euclidean space the Eq. (2.7) describes a second-order cone. In addition

$$\operatorname{Tr}\left(\Lambda-\frac{1}{n}\operatorname{Tr}\Lambda E_n\right)\equiv 0$$

which is a necessary and sufficient condition for the existence of an n-hedral corner with pairwise-orthogonal edges, inscribed in cone (2.8). Taking this into account, as the columns V_j of matrix V we can take unit vectors directed along the edges of this n-hedral corner. Then the matrix R defined by (2.7) possesses properties (2.5) and (2.6). The last equality in (2.6) follows immediately from (2.7) by the nonsingularity of matrix V. It remains to prove the lemma's last assertion. Denoting

$$\Lambda_0=\Lambda-\frac{1}{n}\operatorname{Tr}\Lambda E_n=\operatorname{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n),\quad \lambda_i=\mu_i-\frac{1}{n}\operatorname{Tr}\Lambda$$

we have

$$\varphi_{jj}\equiv V'\Lambda_0 V_j=0,\quad \varphi_{ij}\equiv V_i'V_j=\delta_{ij}\qquad (i,j=1,2,\ldots,n)$$

We set up this system's functional determinant

$$\det\frac{\partial(\varphi_{jj},\varphi_{ij})}{\partial V_j}=\det\begin{Vmatrix}\dfrac{\partial\varphi_{jj}}{\partial V_j}\\[2mm] \dfrac{\partial\varphi_{ij}}{\partial V_j}\end{Vmatrix}=\det\begin{Vmatrix}2V\Lambda_0'\\ \\ V'\end{Vmatrix}=\det\operatorname{diag}(2V',V')\det\operatorname{diag}(\Lambda_0,E_n)\neq 0$$

The latter is true because V, as an orthogonal matrix, is nonsingular and matrix Λ_0 also is nonsingular by virtue of (2.4). According to the theorem on the existence and uniqueness of implicit functions the matrix V, as also the matrix Λ, is continuous and has a continuous derivative in argument t in a neighborhood of the system $V_1,\ldots,V_n$ constructed.

Since a Hermitian matrix can be reduced to a real diagonal matrix by a unitary transformation, we state the following lemma on the expansion of a Hermitian matrix.

Fundamental lemma. A positive-definite Hermitian matrix A whose eigenvalues $\mu_1,\mu_2,\ldots,\mu_n$ satisfy the condition

$$\mu_i\neq\frac{1}{n}\sum_{j=1}^{n}\mu_i\quad (i=1,2,\ldots,n)$$

can be represented as

$$A^{-1}=HH^*,\quad H=(h_1,h_2,\ldots,h_n) \tag{2.10}$$

$$\|h_j\|=\sigma=\sqrt{\frac{1}{n}\operatorname{Tr}A^{-1}},\quad \operatorname{rank}H=\operatorname{rank}A$$

(H is a square matrix), and, if $A(t)$ is continuous and continuously differentiable on $[t_0,T)$, then on this interval the matrix $H(t)$ is continuous and continuously differentiable.

3. Thus, we can examine the perturbed motion of the dynamic system represented by the vector Eq. (2.1).

Theorem 3.1. (On stability). Let a positive-definite Hermitian form

$$V(t,x)=x^*A(t)x \tag{3.1}$$

exist such that:

1°. $A(t_0)=(G_0^{-1})^*G_0^{-1}$ (G_0 is a prescribed constant matrix of class K_Δ^{ω});

2°. $\frac{1}{n}\operatorname{Tr}A^{-1}(t)\leqslant\omega^2(t),\quad \forall t\in[t_0,T)$;

3°. $dV/dt\leqslant 0$ for $\forall t\in[t_0,T)$ (here and later we assume that the derivative of function V with respect to t is taken relative to Eq. (2.1)). Then system (2.1) is stable on interval $[t_0,T)$.

Proof. According to the fundamental lemma the matrix A in form $V(t,x)$ can be represented as $A(t)=[H^{-1}(t)]^*H^{-1}(t)$, where $H(t)$ is a square matrix satisfying the conditions in (2.10), and, in accord with the theorem's condition 1°, $\sigma(t_0)=\omega(t_0)=\omega_0$. The matrix $G(t)\doteq(\omega(t)/\sigma(t))H(t)$ belongs to class K_Δ^{ω}, and $G(t_0)=G_0$. Let $x^\circ(t)$ be some solution of Eq. (2.1), satisfying the condition

$$(G^{-1}(t_0)x^\circ(t_0), G^{-1}(t_0)x^\circ(t_0)) \leqslant \rho^2$$

On the strength of condition 3°, $V(t, x^\circ(t)) \leqslant V(t_0, x^\circ(t^\circ))$ along this solution. Therefore, also allowing for condition 2°, we have

$$(G^{-1}(t)x^\circ(t), G^{-1}(t)x^\circ(t)) = \frac{\sigma^2(t)}{\omega^2(t)} V(t, x^\circ(t)) \leqslant V(t_0, x^\circ(t_0)) \leqslant \rho^2$$

which proves the theorem.

Theorem 3.2. (On instability). Let a positive-definite Hermitian form

$$V(t, x) = x^* A(t) x$$

and an instant $t_1 \in [t_0, T)$ exist such that:

1°. $A(t_0) = (G_0^{-1})^* G_0^{-1}$ (G_0 is a prescribed constant matrix of class $K_\Delta{}^\omega$);

2°. $\mu_{\min}(t_1) > 2\omega^2(t_1)$ ($\mu_{\min}$ is the minimal eigenvalue of matrix A^*A when $t = t_1$);

3°. $dV/dt \geqslant 0$ for $\forall t \in [t_0, t_1]$ and $\forall x \in D$.

Then system (2.1) is unstable on interval $[t_0, T)$.

Proof. Let us assume, to the contrary, that system (2.1) is stable under the fulfillment of the theorem's hypotheses, and, hence, that there exists a matrix $G(t) \in K^\omega{}_\Delta$ such that all solutions of the system, satisfying the condition

$$(G^{-1}(t_0)x(t_0), G^{-1}(t_0)x(t_0)) \leqslant \rho^2 \tag{3.2}$$

on the whole interval $[t_0, T)$, satisfy the condition

$$(G^{-1}(t)x(t), G^{-1}(t)x(t)) \leqslant \rho^2 \tag{3.3}$$

Let $x^\circ(t)$ be a solution of the system, satisfying the condition

$$(G^{-1}(t_0)x^\circ(t_0), G^{-1}(t_0)x^\circ(t_0)) = \rho^2$$

On the strength on the theorem's condition 1° we have $V(t_0, x^\circ(t_0)) = \rho^2$. Let $\nu_{\max}(t_1)$ be the maximal eigenvalue of matrix $G^*(t_1)G(t_1)$. The inequality $\nu_{\max} < 2\omega^2$ holds. Hence, keeping the theorem's condition 2° in mind, we find

$$\mu_{\min}(t_1) > \nu_{\max}(t_1)$$

By virtue of the last relation we obtain

$$G^{-1}(t_1)x^\circ(t_1) \geqslant \frac{\|x^\circ(t_1)\|}{\nu_{\max}(t_1)} > \frac{\|x^\circ(t_1)\|}{\mu_{\min}(t_1)} \geqslant \rho^2$$

which contradicts the stability condition. Hence, the original premise (on the system's stability) is false and the system (2.1) is unstable. There the following stronger theorem holds:

Theorem 3.3. (On instability). Let a positive-definite Hermitian form

$$V(t, x) = x^* A(t) x$$

and an instant $t_1 \in [t_0, T)$ exist such that:

1°. $A(t_0) = (G_0^{-1})^* G_0^{-1}$ (G_0 is a prescribed constant matrix of class $K_\Delta{}^\omega$);

2°. $\frac{1}{n} \operatorname{Tr} A^{-1}(t_1) > \omega^2(t_1)$;

3°. $dV/dt \geqslant 0$ for $\forall t \in [t_0, t_1]$ and $\forall x \in D$.

Then system (2.1) is unstable on interval $[t_0, T)$.

Proof. We show at first that all solutions of (2.1), satisfying the condition

$$V(t_1, x(t_1)) = x^*(t_1)A(t_1)x(t_1) \leqslant \rho^2$$

satisfy the condition

$$V(t_0, x(t_0)) = x^*(t_0)A(t_0)x(t_0) \leqslant \rho^2$$

Let us assume, to the contrary, that there is a solution $x^\circ(t)$ not having such a property. Then by continuity an instant exists such that

$$V(\tau, x^\circ(\tau)) = \rho^2, \; V(t, x^\circ(t)) > \rho^2, \quad \forall t \in [t_0, \tau)$$

In particular, $V(t_0, x^\circ(t_0)) > \rho^2$, but this contradicts the inequality

$$V(t_0, x^\circ(t_0)) \leqslant V(\tau, x^\circ(\tau)) = \rho^2$$

which follows from the theorem's condition 3°. Now we assume that in spite of the theorem's assertion there exists a matrix $G(t)$ of class $K_\Delta{}^\omega$, such that all solutions $x(t)$ satisfying (3.2) satisfy (3.3) on $[t_0, T)$. We introduce into consideration the sets

$$U_G(t) = \{x: (G^{-1}(t)x, G^{-1}(t)x) \leqslant \rho^2\}, \; U_H(t) = \{x: V(t, x) \leqslant \rho^2\}$$

According to the assumptions made (on the system's stability), $U_H(t_1) \subset U_G(t_1)$, but this is impossible since by the theorem's condition 2° the set

$$U_H(t_1) \setminus U_H(t_1) \cap U_G(t_1)$$

is nonempty. Here we used the following lemma: let

$$U_1 = \{x: V_1(t, x) \leqslant \rho^2\}, \; U_2 = \{x: V_2(t, x) \leqslant \rho^2\}$$

where V_1 and V_2 are positive-definite Hermitian forms with matrices A_1 and A_2, respectively; if $\operatorname{Tr} A_1^{-1} > \operatorname{Tr} A_2^{-1}$, then $U_1 \setminus U_1 \cap U_2$ is a nonempty set.

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